BRIEF REPORTS

Density of states of a damped quantum oscillator

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We calculate the density of states of a damped quantum-mechanical harmonic oscillator which is described by a Caldeira-Leggett type model with Ohmic dissipation and a Drude-like cutoff. From the exact expression for the associated partition function, we derive the asymptotic behavior of the density of states using Tauberian theorems. An effective algorithm to evaluate the density of states is presented and examples are given. It is pointed out that the calculated density of states is an experimentally accessible quantity.

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INTRODUCTION

Dissipative quantum systems have been studied quite extensively in the past decade [1]. A particularly simple and completely solvable example of such a system is the damped harmonic oscillator, where a single quantum-mechanical harmonic oscillator is coupled linearly to an infinite set of bath oscillators [2,3]. The coupling is adjusted such as to give a damped dynamics with a phenomenological damping coefficient γ. The reduced partition function Z of this system with Drude regularization is known exactly [1]. It can be obtained in various ways, e.g., by imaginary time path integral methods [4] or stochastic modeling [5], and reads

$$Z(\beta) = \frac{\beta\omega}{4\pi^2} \frac{\Gamma(\beta\lambda_1/2\pi)\Gamma(\beta\lambda_2/2\pi)\Gamma(\beta\lambda_3/2\pi)}{\Gamma(\beta\omega_c/2\pi)} , \quad (1)$$

where $\beta = 1/T$, ω is the frequency of the bare oscillator, and ω_c is a Drude-like frequency cutoff (we choose units such that $k_B = \hbar = 1$). $\Gamma(z)$ is the usual gamma function and λ_i, $i = 1,2,3$, are the roots of the cubic equation

$$\lambda^3 - \omega_c\lambda^2 + (\omega^2 + \gamma\omega_c)\lambda - \omega_c\omega^2 = 0 , \quad (2)$$

which can be obtained exactly, e.g., by means of Cardan's formula [6]. The λ_i are in general complex numbers but have positive definite real parts if $\gamma > 0$.

The partition function $Z(\beta)$ defines the *density of states* $\rho(\varepsilon)$ by the Laplace transform

$$Z(\beta) = \int_0^\infty \rho(\varepsilon)e^{-\beta\varepsilon}d\varepsilon \quad (3)$$

and in principle $\rho(\varepsilon)$ is uniquely determined by the in-

verse transformation

$$\rho(\varepsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(\beta)e^{\beta\varepsilon}d\beta , \quad (4)$$

where $c > 0$ is a real constant. Surprisingly, this quantity—to our knowledge—has not been calculated before. It is the purpose of this Brief Report to calculate $\rho(\varepsilon)$ explicitly for the exactly solvable harmonic oscillator case. This allows one to study quantitatively within a simple model how the discrete spectrum of the undamped oscillator

$$\rho_{(0)}(\varepsilon) = \sum_{n\geq 0} \delta(\varepsilon - (n+\tfrac{1}{2})\omega) \quad (5)$$

is broadened by a finite dissipation $\gamma > 0$. Besides this theoretical interest we will see that the density of states of a damped oscillator is also accessible by experiments.

A direct numerical evaluation of the right-hand side (RHS) of Eq. (4) is rather difficult since the Γ functions appearing in (1) with their complex arguments are strongly oscillating as β is increased. Therefore, we derive some exact results and a series representation of the rhs of Eq. (4) which allows one to evaluate $\rho(\varepsilon)$ without much numerical effort. It turns out that within our approach it is very easy to show, e.g., that the damped system has a unique ground state and in addition a purely absolute continuous spectrum. In general a proof of such properties involves mathematically more sophisticated techniques [3] and has failed so far for more complex quantum systems with dissipation, cf., e.g., [7]. Finally we discuss our results and comment on possible experimental relevance.

EXACT RESULTS

From (1) the free energy F associated with $Z(\beta)$ is trivially given by

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$$F(T) = -T \left\{ \ln \left[\frac{\omega}{4\pi^2 T} \right] + \ln \Gamma \left[\frac{\lambda_1}{2\pi T} \right] + \ln \Gamma \left[\frac{\lambda_2}{2\pi T} \right] \right.$$

$$\left. + \ln \Gamma \left[\frac{\lambda_3}{2\pi T} \right] - \ln \Gamma \left[\frac{\omega_c}{2\pi T} \right] \right\} . \qquad (6)$$

By means of an asymptotic expansion [8] of the function $\ln \Gamma(z)$ for large arguments z we find the asymptotic behavior

$$F(T) \sim \varepsilon_0 - \frac{\gamma \pi}{6\omega^2} T^2 , \quad T \to 0 , \qquad (7)$$

where the ground state energy ε_0 is given by

$$\varepsilon_0 = \frac{1}{2\pi} \{ \lambda_1 \ln(\omega_c/\lambda_1) + \lambda_2 \ln(\omega_c/\lambda_2) + \lambda_3 \ln(\omega_c/\lambda_3) \} . \qquad (8)$$

With the help of formula (6.60) of Ref. [1] ε_0 can also be written in the alternative form

$$\varepsilon_0 = \frac{\omega}{2\pi} \int_0^\infty d\alpha \ln \left[\frac{1 + \alpha^2 + \gamma \omega_c \alpha / (\omega_c \omega + \alpha \omega^2)}{\alpha^2} \right] . \qquad (9)$$

This shows that the ground state energy ε_0 is an increasing function of the dissipation γ and ω_c. The T^2 correction to $F(T)$ at low temperatures is characteristic for a system with Ohmic dissipation. It leads to a linear specific heat as $T \to 0$, indicating that the spectrum has no gap above the ground state. To study this in more detail we investigate the asymptotic behavior of $\rho(\varepsilon)$ for $\varepsilon \to \varepsilon_0 +$ and $\varepsilon \to \infty$. We define functions $\Sigma(\beta)$ and $\mathcal{R}(x)$, $x > 0$, by

$$Z(\beta) = e^{-\beta F} =: e^{-\beta \varepsilon_0} [1 + \Sigma(\beta)] \qquad (10)$$

and

$$\Sigma(\beta) =: \int_0^\infty \mathcal{R}(x) e^{-\beta x} dx . \qquad (11)$$

This implies that

$$\rho(\varepsilon) = \delta(\varepsilon - \varepsilon_0) + \mathcal{R}(\varepsilon - \varepsilon_0) \theta(\varepsilon - \varepsilon_0) , \qquad (12)$$

where $\delta(\varepsilon - \varepsilon_0)$ is the Dirac delta distribution centered at ε_0 and $\theta(x) = 1$ if $x > 0$ and zero otherwise. From (7) and since $Z(\beta) \sim Z_{cl}(\beta) = 1/\beta\omega$ in the classical limit $\beta \to 0$ we may conclude that the function $\Sigma(\beta)$ vanishes like $\gamma \pi / 6\omega^2 \beta$ for $\beta \to \infty$ while in the classical limit $\beta \to 0$ it diverges like $1/\beta\omega$. We now apply Tauberian's theorem [9] which determines the behavior of $\mathcal{R}(x)$ for small arguments where

$$\mathcal{R}(x) \sim \frac{\gamma \pi}{6\omega^2} , \quad x \to 0+ , \qquad (13)$$

and in the limit $x \to \infty$, where

$$\frac{1}{x} \int_0^x \mathcal{R}(x') dx' \sim \frac{1}{\omega} , \quad x \to \infty \qquad (14)$$

(here $x = \varepsilon - \varepsilon_0$; see above). In the next section we show that $\mathcal{R}(x)$ is absolutely continuous in $(0, \infty)$. Therefore, as expected on physical grounds [3], $\rho(\varepsilon)$ displays a

unique ground state with energy ε_0 and in addition a purely absolute continuous (in fact $\mathcal{C}^\infty$) spectrum (ε_0, ∞).

RESIDUUM ANALYSIS

Let us now apply the residue theorem in order to calculate the rhs of Eq. (4). First we note that either one of the roots λ_i of Eq. (2) is real while the other two are complex conjugate with nonvanishing imaginary parts or they are all real. In the first case we call the oscillator *underdamped*, otherwise *overdamped*. It turns out that in the limit $\omega_c \gg \omega, \gamma$ this corresponds to the standard classical definition $\gamma/2 < \omega$ or $\gamma/2 > \omega$, respectively [6]. The characterizing feature of the underdamped case is that $\rho(\varepsilon)$ is an oscillating function of ε; see below.

We first discuss the underdamped case. The partition function $Z(\beta)$ in Eq. (1) has a simple pole at $\beta_0 = 0$ with residue $R_0 = 1/\omega$ and further poles $\beta_{n,i} = -2\pi n/\lambda_i$, $n \in \mathbb{N}$, $i = 1,2,3$. Let λ_1 and $\lambda_2 = \lambda_1^*$ be the complex conjugate roots with nonvanishing imaginary parts. The poles $\beta_{n,i}$ are located on different axes in the complex half plane $\mathrm{Re}\beta < 0$; see Fig. 1. We now deform the integration path in the line integral on the rhs of Eq. (4) to $\mathcal{C}'$ and add line integrals along $\mathcal{C}_2$ and $\mathcal{C}_3$ which give no contribution according to Cauchy's integral theorem. In the usual way we then arrive at

$$\rho(\varepsilon) = \frac{1}{\omega} + \sum_{n=1}^\infty \sum_{i=1}^3 R_{n,i} \exp(-2\pi n\varepsilon/\lambda_i) , \qquad (15)$$

where the residues $R_{n,i} \equiv \mathrm{Res}(Z(\beta), \beta_{n,i})$ are easy to calculate since all poles $\beta_{n,i}$ are simple. We find

$$R_{n,1} = \frac{\omega(-1)^{n+1}}{\lambda_1^2(n-1)!} \frac{\Gamma(-n\lambda_2/\lambda_1)\Gamma(-n\lambda_3/\lambda_1)}{\Gamma(-n\omega_c/\lambda_1)} \qquad (16)$$

and analogous expressions for $R_{n,2}$ and $R_{n,3}$ by a permutation of indices. Since $\mathrm{Im}\lambda_1 \neq 0$ and $\mathrm{Re}\lambda_1 > 0$ the function $\rho(\varepsilon)$ is oscillating with exponentially decreasing amplitudes (remember that $\lambda_2 = \lambda_1^*$ and λ_3 is real).

Note that the two axes corresponding to the complex conjugate roots move towards the negative real axis as the damping coefficient γ is increased. The negative real
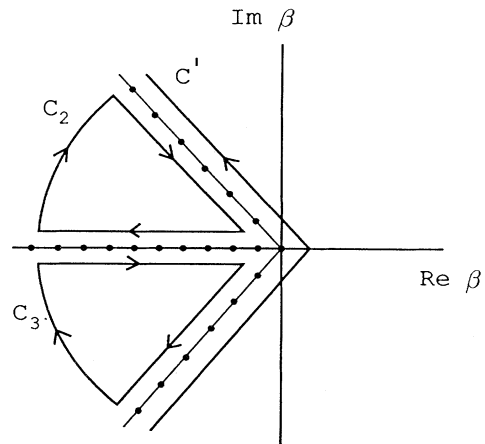

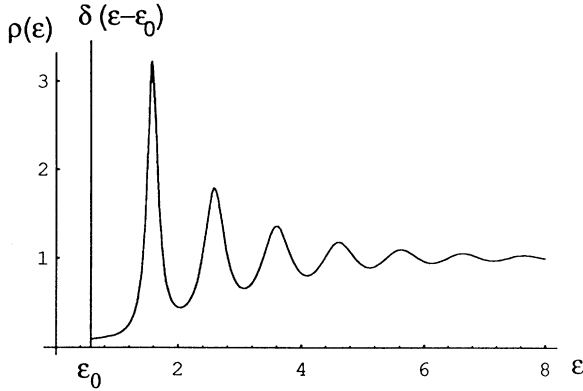FIG. 1. Integration paths in the complex β plane for an underdamped oscillator with $0 < \gamma < \gamma_c$.

FIG. 2. Density of states $\rho(\varepsilon)$ for an underdamped oscillator with $\omega=1$, $\gamma=0.2$, and $\omega_c=10$. Of course, the δ peak has been added by hand.

FIG. 3. Density of states $\rho(\varepsilon)$ for an overdamped oscillator with $\omega=1$, $\gamma=5$, and $\omega_c=10$.

axis is reached for a value $\gamma \equiv \gamma_c$ which separates the underdamped from the overdamped case. If all λ_i are real, expression (15) for $\rho(\varepsilon)$ together with (16) still holds (for simplicity we assume that all poles are different and thus simple). Since all λ_i are real, $\rho(\varepsilon)$ is now monotonically decreasing.

From (15) we see that $\rho(\varepsilon) \sim 1/\omega$ for $\varepsilon \to \infty$, in accordance with (14). In order to show that $\rho(\varepsilon)$ is an absolutely continuous function of ε in (ε_0, ∞) we apply the asymptotic expansion [8] of $\ln\Gamma(z)$ for large arguments z to the Γ functions appearing on the rhs of Eq. (16). Using relations between the λ_i [6] we find the asymptotic estimate

$$\left| R_{n,1} \exp\left[\frac{-2\pi n \varepsilon}{\lambda_1} \right] \right| \lesssim \left| \frac{1}{\lambda_1} \exp\left[\frac{-2\pi n (\varepsilon - \varepsilon_0)}{\lambda_1} \right] \right|$$

(17)

for large n (similarly for $i=2,3$). Since the λ_i have positive definite real parts the terms in the sum of Eq. (15) decrease exponentially fast for large n if $\varepsilon > \varepsilon_0$. Thus the series and also the series of the derivatives with respect to ε are absolutely and uniformly convergent in (ε_0, ∞). Therefore the density of states $\rho(\varepsilon)$ is a continuously differentiable (in fact $\mathcal{C}^\infty$) and hence absolutely continuous function of ε in (ε_0, ∞). This implies that apart from the ground state there are no further embedded eigenstates, cf. [3].

DISCUSSION

To give an example we use Eq. (15) to evaluate $\rho(\varepsilon)$ for an underdamped ($\omega=1$, $\gamma=0.2$, $\omega_c=10$) and an overdamped case ($\omega=1$, $\gamma=5$, $\omega_c=10$); see Figs. 2 and 3, respectively. The behavior of $\rho(\varepsilon)$ is as expected from the results above: The *ground state* remains as a separate δ function centered at the ground state energy ε_0 which is shifted *upwards* compared with the undamped case where $\varepsilon_0=0.5$ for $\omega=1$; cf. Eq. (9) ($\varepsilon_0 \approx 0.57$ in the underdamped, $\varepsilon_0 \approx 1.95$ in the overdamped case). Evidently the *excited* states of the bare oscillator are broadened by
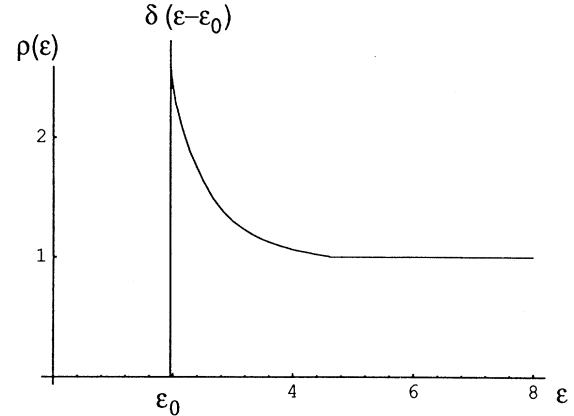
the damping and the discreteness of the original spectrum completely disappears at large energies where $\rho(\varepsilon \to \infty)=1/\omega$. Note that this is just the *average* density of states consistent with the undamped case where the discrete levels appear regularly with distance ω (i.e., the steps of size 1 in the integrated density of states are replaced by a straight line). At low energies $\varepsilon \to \varepsilon_0+$ the *continuous* part of $\rho(\varepsilon)$ starts with a finite value $\gamma\pi/6\omega^2$ (≈ 0.1 in the underdamped, ≈ 2.6 in the overdamped case), as expected from (13). The positions of the peaks in the underdamped case and the scale of the exponential decay follow from the imaginary and real parts of the λ_i.

Apart from the intrinsic interest in studying the evolution from a discrete point spectrum to a continuum by dissipation, the density of states also determines the absorbed power in an experiment where the oscillator is driven by an external perturbation. Indeed, the absorption of an external field with frequency ω_0 will be proportional to $\rho(\varepsilon_0+\omega_0)$. Experiments of this type were performed in a current-biased Josephson junction where the damped oscillator consists of the phase difference of the order parameter on both sides of the junction in the metastable zero voltage state [10]. The phase difference may be represented as a quantum-mechanical particle in a cubic potential, subject to (weak) dissipation; cf. also [1]. In these experiments the escape rate from the zero voltage state was measured when a microwave current at fixed power P and frequency ω_0 was applied. Using a device of the same type at small dc current where the escape is negligible it should be possible to observe the absorbed microwave power when the frequency ω_0 is changed. Thus in principle it should be possible to measure the density of states calculated here by an experiment in which a current-biased junction is driven by a microwave field.

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